A New Approach for Identification of Physical Matrices by Modal Testing

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ABSTRACT

An approach for experimental identification of the mass, the damping and the stiffness matrices is proposed with using the modal parameters obtained by modal testing. A new constraining condition is introduced for reasonable identification of the modal parameters for this purpose. Validity of the proposed approach is verified experimentally.

1. INTRODUCTION

Experimental identification of dynamic characteristics of a structure is divided roughly into the following two kinds. One is identification of the modal parameters which is called modal testing or modal analysis, and the other is identification of the physical matrices (the mass, the damping and the stiffness matrices), which is called sometimes as system identification.

Most of experimental identifications now used in actual fields belong to the former, and many kinds of softwares and apparatus for the former are now on the market. On the other hand, concerning the latter, direct approaches and techniques for identification of the physical matrices from the frequency response function (FRF) obtained by the vibration testing have been studied vigorously in Japan, because it is useful for various vibration analyses such as combination of the experimental analysis with the finite element method to compose the mathematic model, i.e., the equation of motion, directly by the vibration testing. But some difficulties exist in this direct experimental identification of the physical matrices. For example, accuracy of identification is easily deteriorated by errors in experimental values of FRF used as the input for identification. The present paper is a report on a study for solving these difficulties in direct identification of the physical matrices, and the content is as follows.

A new approach is proposed for identification of the physical matrices (the mass, the damping and the stiffness matrices in the spatial equation of motion) with using the modal parameters obtained by modal testing. The modal parameters are determined by the multi-degrees of freedom curve fitting technique in the frequency domain with the frequency response function (FRF) measured in the vibration test. A new constraining condition is introduced for reasonable identification under the assumption of general viscous damping. The reciprocal of variance of the measured value of FRF is adopted in the process of identification as the weight of the square error from the viewpoint of the statistic theory.

The method of identification of the physical matrices is explained. The reasonable physical matrices are obtained with the modal parameters, when the new constraining condition is used. But the physical matrices cannot be obtained when this constraining condition is ignored. Effectiveness of the new approach proposed in the present paper is verified experimentally by the vibration test of a steel bar.

2. THEORETICAL BASIS

2.1 Relation between Physical Matrices and Modal Parameters

The equation of motion of the multiple degrees of freedom system under the assumption of general viscous damping is

$$[D] \{ \ddot{q} \} + [E] \{ q \} = \{ p \}$$

(1)

where

$$[D] = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \quad [E] = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}$$

(2)

$$\{ q \} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \{ p \} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Here, $[M],[C]$ and $[K]$ are the mass, the damping and the stiffness matrices respectively, and are called as the...
Comparing Eqs.(11) and (12) with each other, relation between the physical matrices and the modal parameters becomes as follows.

\[
[M] = [\phi \Lambda \phi^T + \phi^* \Lambda^* \phi^*]^T
\]
\[
[K] = -[\phi \Lambda^{-1} \phi^T + \phi^* \Lambda^{-1} \phi^*]^T
\]
\[
[C] = -[M][\phi \Lambda^2 \phi^T + \phi^* \Lambda^2 \phi^*][M]
\]  

2.2 Constraining Condition to Natural Modes

Comparing Eq.(11) with Eq.(12) again, we see easily that the natural modes must satisfy the following condition.

\[
Re[\phi \phi^T] = [0]
\]  

where \( Re[ \cdot ]\) means the real part of the complex number. Equation (14) is different of course from the property of orthogonality of the natural modes. Next, the physical meaning of Eq.(14) is explained.

The frequency response function \( G_{ij}(\omega) \) is obtained from the equation of motion (1) (see Appendix 1).

\[
G_{ij}(\omega) = \sum_{r=1}^{n} \frac{U_{ri} \phi_{r} \phi_{ij} + U_{ri} - jV_{ri}}{j(\omega^2 - \omega_d^2) + \sigma_r} \]  

where

\[
U_{rij} = \phi_{ri} \phi_{r} \phi_{ij}
\]

The response displacement due to the unit impulse force is obtained as follows by inverse Fourier transform of Eq.(15).

\[
h_{ij}(t) = 2 \sum_{r=1}^{n} e^{-\sigma_r t} (U_{rij} \cos \omega_d t + V_{rij} \sin \omega_d t)
\]  

As the impulse response displacements must be zero at \( t = 0 \), \( h_{ij}(t = 0) = 0 \) (\( i = 1 \sim n, j = 1 \sim n \)). Considering the reciprocal property \( U_{rij} = U_{rij} \), this initial condition of Eq.(17) becomes

\[
\sum_{r=1}^{n} U_{rij} = 0 \quad (i = 1 \sim n, j = 1 \sim i)
\]  

On the other hand, rewriting Eq.(14),

\[
\sum_{r=1}^{n} Re[\phi_{ri} \phi_{r}] = 0 \quad (i = 1 \sim n, j = 1 \sim i)
\]  

This problem can be solved as follows. As a matter of fact, the residues \( (U_{rij} \) and \( V_{rij} \) are not independent with one another and have the mutual relation represented with Eq.(18) or (22). We can see easily that the number of Eq.(18) or (22) is \( n(n + 1)/2 \). So, taking Eq.(18) or (22) as the additional constraining condition in the experimental modal analysis, the number of unknown values of the modal parameters becomes 2\( n(n + 1)/2 \) = 3\( n(n + 1)/2 \). This number is equal to that of the physical matrices. This is the reason why the authors insist that this new constraining condition is necessary and indispensable for the correct experimental modal analysis.
Substituting Eq.(16) into Eq.(19), Eq.(18) is obtained. This means that Eq.(14) is the constraining condition which explains the self-evident physical phenomenon that the initial value of impulse response displacement is zero.

This constraining condition is found newly in the present report, and the modal parameters have been determined without considering this condition in all usual experimental modal analysis. So, correct modal parameters are not obtained in the usual curve fitting. Moreover, this constraining condition is indispensable in the experimental modal analysis for determination of the correct physical matrices.

In the actual experimental modal analysis, only the limited number of FRF between the exciting point \( f \) and the response point \( j \) is given as the input data. So, Eq.(18) cannot be used directly as the constraining condition, and it is transformed to a usable form by the following process.

Rewriting both of the subscripts \( i \) and \( j \) to \( f \) in Eq.(16), the natural mode at the exciting point \( f \) can be obtained as follows.

\[
\phi_{rf} = \sqrt{U_{rff} + j\bar{V}_{rff}} \quad (r = 1 \sim n) \tag{20}
\]

Rewriting the subscript \( j \) to \( f \) in Eq.(16), and substituting Eq.(20) to it,

\[
\phi_{rf} = \sqrt{U_{rff} + j\bar{V}_{rff}} \quad (r = 1 \sim n) \tag{21}
\]

Substituting Eq.(21) and \( \phi_{rf} \) obtained directly from Eq.(21) \((i \rightarrow j)\) into Eq.(19),

\[
\sum_{r=1}^{n} \frac{U_{rff}(U_{rfj} + V_{rfj}) + V_{rff}(U_{rfj} + V_{rfj})}{U_{rff} + j\bar{V}_{rff}} = 0 \quad (i = 1 \sim n, j = 1 \sim n) \tag{22}
\]

Equation (22) is obtained more simply from Eq.(16) as follows

\[
U_{rfj} + jV_{rfj} = \phi_{rf}\phi_{rf} = \frac{(\phi_{rf}\phi_{rf} + \phi_{rf}\phi_{rf})}{\phi_{rf}} = \frac{(U_{rfj} + jV_{rfj})(U_{rfj} + jV_{rfj})}{U_{rfj} + jV_{rfj}} \tag{23}
\]

The real part of Eq.(23) is equal to Eq.(22).

The physical matrices cannot not be determined from the modal parameters correctly with using only Eq.(13), because the numbers of the unknown values differ as follows with each other. Namely, the total number of the modal parameters of the \( n \) degrees of freedom system is \( 2n(n+1) \), for the number of \( \omega_i \) and \( \sigma_i \) is \( 2n \), and the number of \( U_{ri}\) and \( V_{ri}\) is \( 2n^2(n-1 \sim n, i = 1 \sim n) \) on the other hand, the total number of unknown elements of three physical matrices (the mass, the damping and the stiffness matrices) of the \( n \) degrees of freedom system is \( 3n(n+1)/2 \), for all physical matrices are symmetric. Thus, the number of unknown values of the modal parameters is larger as \( n(n+1)/2 \) than that of the physical matrices.

3. DETERMINATION OF MODAL PARAMETER

The modal parameters should be determined so that the following error function \( S \) which is explained as the weighted residual sum of squares of FRF becomes minimum.

\[
S = \sum_{i=1}^{N} \sum_{j=1}^{n} \left( \left| W_{ij}(\omega_i) \left( G_{ij}(\omega_i) - V_{ij}(\omega_i) \right) \right|^2 + W_{ij}(\omega_i) \left( G_{ij}(\omega_i) - V_{ij}(\omega_i) \right)^2 \right) \tag{21}
\]

where \( G \) is the mathematic expression of FRF of Eq.(15) and is the function of unknown modal parameters. \( A_{rj} \) and \( A_{ij} \) are the experimental FRF obtained by a vibration test, \( \omega_i (i = 1 \sim N) \) is the angular frequency point at which the experimental FRF is sampled. The subscripts \( R \) and \( J \) denote the real and the imaginary parts of complex number respectively. \( m \) is the number of the referenced FRF for identification. \( W_{ij} \) and \( W_{ij} \) are the weighting coefficients which are equal to the inverses of the variances in case that \( A_{rj} \) and \( A_{ij} \) have Gaussian distributions with random errors.

The author propose to adopt Eq.(22) as the constraining condition. Equation (22) is rewritten simply for convenience as

\[
g_j(x) = 0 \quad (i = 1 \sim q = n(n+1)/2) \tag{25}
\]

where \( x \) represents the unknown modal parameters. The modal parameters are determined so that Eq.(24) becomes minimum under the constraining condition of Eq.(25) by the gradient projection method explained in Appendix 2.

4. ILLUSTRATIVE EXAMINATION

Validity of the proposed method is examined experimentally with a steel bar of a rectangular section shown in Fig.1. Point 1 of this bar is excited under the free condition, and responses of points 1 \sim 4 are measured at every 5 Hz from 440 Hz to 1680 Hz. Using FRF \((A_{rj} \) and \( A_{ij})\) obtained by this vibration test as the input data, the modal parameters are determined under the assumption that the freedom of this bar is 4 degrees. Next, the matrices \([D]\) and \([E]\) are determined from Eqs.(8) and (9).

It is obvious from the definition of Eq.(2) that the upper, right quarter of \([D]\), the lower, left quarter of \([D]\), and the negative of the lower, right quarter of \([E]\) must be...
Table 1 Physical Matrices Identified by the Usual Method

[D] matrix

| .12E - 02 | -.89E + 00 | -.70E + 00 | .56E + 00 | .32E - 03 | .94E - 03 | .78E - 03 | -.49E - 03 |
| -.89E + 00 | -.64E + 01 | -.48E + 01 | .34E + 01 | .90E - 03 | .56E - 02 | .45E - 02 | -.49E - 02 |
| -.70E + 00 | -.48E + 01 | -.36E + 01 | .27E + 01 | .76E - 03 | .45E - 02 | .34E - 02 | -.24E - 02 |
| .56E + 00 | .34E + 01 | .27E + 01 | -.19E + 01 | -.16E - 03 | -.24E - 02 | -.20E - 02 | .13E - 02 |

| .32E - 03 | .90E - 03 | .76E - 03 | -.46E - 03 | .21E - 07 | .19E - 06 | .14E - 06 | -.77E - 07 |
| .94E - 03 | .56E - 02 | .45E - 02 | -.24E - 02 | .19E - 06 | .12E - 05 | .01E - 06 | -.52E - 06 |
| .78E - 03 | .45E - 02 | .40E - 02 | -.20E - 02 | .14E - 06 | .91E - 06 | .68E - 06 | -.40E - 06 |
| -.49E - 03 | -.25E - 02 | -.21E - 02 | .13E - 02 | -.77E - 07 | -.52E - 06 | -.40E - 06 | .22E - 06 |

[E] matrix

| .30E + 04 | .15E + 05 | .86E + 04 | -.95E + 04 | .30E + 00 | .35E + 01 | .25E + 01 | -.14E + 01 |
| .15E + 05 | .86E + 05 | .64E + 05 | -.50E + 05 | .16E + 01 | .12E + 02 | .68E + 01 | -.50E + 01 |
| .86E + 04 | .64E + 05 | .75E + 05 | -.55E + 05 | .11E + 01 | .81E + 01 | .60E + 01 | -.35E + 01 |
| -.95E + 04 | -.50E + 05 | -.55E + 05 | .41E + 05 | -.93E + 00 | -.71E + 01 | -.32E + 01 | .31E + 01 |

| .30E + 00 | .16E + 01 | .11E + 01 | -.93E + 00 | -.30E - 03 | -.78E - 03 | .66E - 03 | .42E - 03 |
| .35E + 01 | .12E + 02 | .81E + 01 | -.71E + 01 | -.78E - 03 | -.44E - 02 | -.36E - 02 | .20E - 02 |
| .25E + 01 | .86E + 01 | .60E + 01 | -.52E + 01 | .66E - 03 | -.36E - 02 | -.33E - 02 | .18E - 02 |
| -.14E + 01 | -.50E + 01 | -.35E + 01 | .31E + 01 | .42E - 03 | .20E - 02 | .18E - 02 | .12E - 02 |

Table 2 Physical Matrices Identified by the Proposed Method

[D] matrix

| .43E + 00 | .18E + 01 | .99E + 00 | -.52E + 00 | .33E - 03 | .11E - 02 | .92E - 03 | -.57E - 03 |
| .18E + 01 | .22E + 01 | -.70E + 00 | .15E + 00 | .11E - 02 | .73E - 02 | .59E - 02 | -.34E - 02 |
| .99E + 00 | -.70E + 00 | -.24E + 01 | .12E + 01 | .92E - 03 | .59E - 02 | .51E - 02 | -.28E - 02 |
| -.52E + 00 | -.15E + 00 | -.12E + 01 | -.64E + 00 | -.57E - 03 | -.34E - 02 | -.28E - 02 | .18E - 02 |

| .33E - 03 | .11E - 02 | .92E - 03 | -.57E - 03 | -.14E - 14 | .62E - 15 | -.83E - 15 | .50E - 15 |
| .11E - 02 | .73E - 02 | .59E - 02 | -.34E - 02 | -.62E - 15 | .14E - 14 | .74E - 15 | -.72E - 15 |
| .92E - 03 | .59E - 02 | .51E - 02 | -.28E - 02 | -.83E - 15 | .74E - 15 | .28E - 15 | .54E - 16 |
| -.57E - 03 | -.34E - 02 | -.28E - 02 | .18E - 02 | -.50E - 15 | -.72E - 15 | .54E - 16 | -.34E - 15 |

[E] matrix

| .91E + 04 | .16E + 05 | .93E + 04 | -.10E + 05 | -.43E - 07 | .26E - 07 | -.34E - 07 | .15E - 07 |
| .16E + 05 | .77E + 05 | .71E + 05 | -.56E + 05 | -.57E - 07 | .15E - 07 | -.21E - 07 | .20E - 07 |
| .93E + 04 | .71E + 05 | .83E + 05 | -.61E + 05 | -.56E - 07 | -.26E - 07 | .59E - 08 | .25E - 07 |
| -.10E + 05 | -.56E + 05 | -.61E + 05 | .46E + 05 | .55E - 07 | .54E - 08 | .10E - 07 | -.23E - 07 |

| -.43E - 07 | -.57E - 07 | -.56E - 07 | .55E - 07 | -.33E - 03 | -.11E - 02 | -.92E - 03 | .60E - 03 |
| -.26E - 07 | -.15E - 07 | .54E - 08 | .54E - 08 | -.11E - 02 | -.73E - 02 | -.59E - 02 | .34E - 02 |
| -.31E - 07 | -.21E - 07 | .59E - 08 | .10E - 07 | -.92E - 03 | -.59E - 02 | -.51E - 02 | .28E - 02 |
| .15E - 07 | .20E - 07 | .25E - 07 | -.23E - 07 | .60E - 03 | .34E - 02 | .28E - 02 | -.18E - 02 |

equal with one another, and that the lower, right quarter of [D], the upper, right quarter of [E] and the lower, left quarter of [E] must vanish.

Table 1 is the results of [D] and [E] determined from the modal parameters which are obtained by the conventional method with only Eq.(24) without using the proposed constraining condition of Eq.(22). [D] and [E] in Table 1 do not satisfy the above requirements, and it is evident from this that the correct physical matrices cannot be obtained by the conventional method.
On the other hand, Table 2 is the results of $[D]$ and $[E]$ determined from the modal parameters which are obtained by the new method proposed in the present report with use of Eq. (22) as the constraining condition. $[D]$ and $[E]$ in Table 2 satisfy the above requirement well, which proves that the correct physical matrices can be obtained by the proposed method.

The solid line in Fig. 2 is the experimental value of FRF (driving point dynamic compliance of the point 4) of the bar in Fig. 1, and three natural modes exist between 410 Hz and 1680 Hz. The chain line in Fig. 2 is the reconstructed FRF from the modal parameters determined by the proposed method under the assumption that the freedom of this bar is 3 degrees. The chain line differs from the solid line near the anti-resonance points. The cause of this difference is that the residual term which represents effect of higher modes is neglected in Eq. (15) used in the proposed method. While, the dotted line in Fig. 2 is the reconstructed FRF from the modal parameters determined by the proposed method under the assumption that the freedom of this bar is 4 degrees. The dotted line agrees well with the solid line, because the additional one mode absorbs skillfully effect of higher modes. This represents that the degrees of freedom assumed in the proposed method should be larger than the number of natural modes contained in the adopted frequency range.

5. CONCLUSIONS

A new approach for determining the physical matrices through the experimental modal analysis is proposed. It becomes clear that a new constraining condition is necessary for correct identification of modal parameters by experimental modal analysis, and that this new constraining condition is indispensable in order to make determination of the physical matrices possible. Validity of the proposed approach is proved experimentally with using a steel bar.

6. ACKNOWLEDGMENT

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\[ \{ Y \} = [\Psi] [j \omega I + [\varepsilon]]^{-1} [\Psi]^T \{ P \} \]  

Substituting the lower part of Eqs. (2), (7) and (9) into Eq. (A6), and taking only its right, upper quarter, the relation between the exciting force \( \{ F \} \) and the response \( \{ X \} \) becomes

\[ \{ X \} = \delta [j \omega I - [\Lambda]] [\delta]^T \{ F \} \]  

Substituting Eq. (6) into \( [\Lambda] \) of Eq. (A7), and taking only its line \( i \), row \( j \) element, the frequency response function (dynamic compliance) \( G_{ij}(\omega) = X_i / F_j \) between the points \( i \) and \( j \) becomes Eq. (15).

In order to determine the physical matrices from the modal parameters, the same number of the natural modes as the degrees of freedom of the system must be taken into account in the experimental modal analysis. So \( r = 1 \sim n \) in Eq. (15), and the residual terms are not adopted.

**Appendix 2**

As Eqs. (24) and (25) are the nonlinear functions of \( x \), the problem is converted into linear one firstly by the method of steepest descent. Adopting only the first terms of the Taylor expansions of Eqs. (24) and (25),

\[ S(x + \delta x) = S(x) + (\nabla S(x))^T \{ \delta x \} \]  

\[ g(x + \delta x) = g(x) + (\nabla g(x))^T \{ \delta x \} \]  

where \( \{ \nabla S \} \) and \( \{ \nabla g \} \) represent the sensitivities of \( g \) and \( S \) respectively as follows.

\[ \{ \nabla S \} = (\partial S / \partial x_1, \ldots, \partial S / \partial x_q) \]  

\[ \{ \nabla g \} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_q} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_q}{\partial x_1} & \cdots & \frac{\partial g_q}{\partial x_q} \end{bmatrix} \]  

In order that \( \{ \delta x \} \) does not be too large,

\[ \{ \delta x \}^T \{ \delta x \} = \varepsilon^2 \]  

where \( \varepsilon \) is a given small value. Using Eqs. (A8) and (A9), the problem becomes to minimize the linear function \( \{ \Delta S(x) \}^T \{ \delta x \} \) under the linear conditions that

\[ \{ \nabla g \} \{ \delta x \} = 0 \]

and Eq. (A12). This problem becomes minimization of the following function \( H \) by introducing Lagrange multipliers \( \lambda \) and \( \beta \).

\[ H = \{ \nabla S \}^T \{ \delta x \} + \lambda \{ \nabla g \} \{ \delta x \} + \beta (\{ \delta x \}^T \{ \delta x \} - \varepsilon^2) \]  

Differentiation of Eq. (A14) with \( \{ \delta x \} \) must vanish in order that \( H \) takes the minimum value, so
\[ \{\nabla S\} + [\nabla g]^T \{\lambda\} + 2\beta \{\delta x\} = 0 \quad (A15) \]

Substituting the solution \{\delta x\} of Eq (A15) into Eq (A13)

\[ [\nabla g][\{\nabla S\} + [\nabla g]^T \{\lambda\}] + \{0\} = \{0\} \quad (A16) \]

From Eq (A16) Lagrange multiplier \{\lambda\} is

\[ \{\lambda\} = -([\nabla g][\nabla g]^T)^{-1} [\nabla g][\nabla S] \quad (A17) \]

Substituting Eq (A17) into Eq (A15)

\[ \{\delta x\} = -([I] + [\nabla g]^T ([\nabla g][\nabla g]^T)^{-1} [\nabla g]) \frac{\{\nabla S\}}{2\beta} \quad (A18) \]

Lagrange multiplier \beta is decided from Eqs (A12) and (A18). Thus, the solution \{x\} + \{\delta x\} is obtained, if the initial value of \{x\} is given. This initial value can be given easily by some simple experimental modal analysis.