Three-layered damped beam element for forced vibration analysis of symmetric sandwich structures with a viscoelastic core

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**Abstract**

The numerical implementation of Mead and Markus's two sets of differential equations of motion governing the damped forced vibration of three-constrained-layer sandwich beam requires \(C^2\)-basis functions or the mixed formulation. To resolve this problem, a damped beam element for three-layered symmetric straight damped sandwich structures is derived according to the virtual work principle, in which both the virtual kinetic and strain energies are expressed in terms of the lateral displacement and the transverse shear strain of a core layer. Because the forced vibration equations of three-constrained-layer damped beam are equipped with three pairs of boundary conditions, the rotation of the mid-surface which is directly derived from the lateral displacement is added for the damped beam element to have three degrees of freedom per node. The shape functions are analytically derived using the compatibility relation between the lateral displacement and the transverse shear strain. The validity of the proposed beam element is verified through the benchmark experiments, and furthermore the DOF-efficiency is justified through the comparison with Nastran 3-D solid element.

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1. Introduction

The suppression of the structural vibration and noise has been a great challenging subject in various engineering fields during several decades, because a dynamic system or its components with insufficient damping may, but frequently, produce the significantly high vibration which results in the unexpected noise and dynamic instability. According to the very intensive and continuing research efforts, a lot of passive and active devices for reducing the structural vibration have been introduced, and some of them are successfully and widely adopted for a variety of engineering applications. Viscoelastic material among the damping materials employed for such devices is widely used to dissipate the structural vibration energy. The high-damped viscoelastic film coated on the critical structure members of submarine for reducing the underwater noise and the viscoelastic material layer introduced into the cylindrical metallic gun tubes for reducing the structural dynamic deflection become the representative examples [1,2].

In particular, the viscoelastic layer inserted between the metallic layers exhibits the significantly high damping effect, called the constrained-layer damping [3,4], according to the high shear deformation of the core viscoelastic layer. The structural vibration of a constrained-layer damping sandwich is characterized by a combination of two distinct deformation modes, the oscillating flexural bending deformation of two metallic faces and the alternating distortional shear deformation of a core viscoelastic layer. Then, the transverse shear strain of the viscoelastic member which is induced by the oscillating flexural bending of the metallic members produces the transverse shear stress with the phase lag within the viscoelastic member. As a result, the oscillating vibratory energy of the sandwich structure is dissipated via the hysteretic loss of the viscoelastic member.

According to our literature survey, the constrained-layer damping has been continuously and intensively studied since the late 1950s, and most of them were motivated by RKU (Ross–Kerwin–Ungar) theory of Ross et al. [5]. They laid down the basic mathematical framework for the viscoelastic constrained-layer sandwich beam and derived the effective, complex and flexural stiffness for the beam section. Since then, based on RKU theory, DiTaranto [4] and Mead and Markus [6] derived the six-order differential equations governing the natural frequencies, the associated composite loss factors and the forced vibration of three-constrained-layer damped beams by introducing the complex shear modulus. These equations are thought as an extension of Euler beam theory to the viscoelastic laminated beam-like structures so that the problem domain is reduced to the one-dimensional neutral axis of structures [7–9]. Thereafter, the
extensive research efforts have been made by the subsequent investigators [10–13], in order to refine the earlier theories by including the additional damping effects due to the extensional/compressive deformations and the rotary inertia and to apply the earlier works to various damped multi-layered structural dynamic problems [2,14–16].

In particular, Sainsbury and Zhang [17] introduced a new more accurate and efficient Galerkin element with eight DOFs for unsymmetrical three-layer damped sandwich beam in which the displacement compatibility over the entire interface between the damping and elastic layers is taken into consideration by combing the conventional polynomial shape functions with Galerkin orthogonal functions to ensure a conforming element and guarantee good accuracy. Trindade et al. [18] proposed an electrically coupled beam element with eight DOFs to handle hybrid active–passive multilayer sandwich beam structures, consisting of a viscoelastic core sandwiched between layered piezoelectric faces, where the frequency-dependence of the viscoelastic material is handled through the anelastic displacement fields (ADF) model. Galucio et al. [19] presented a finite element formulation and an 8-DOF damped sandwich beam for transient dynamic analysis of sandwich beams with embedded viscoelastic material. They used a four-parameter fractional derivative model to describe the frequency-dependence of the viscoelastic layer.

Shorter [20] introduced a spectral finite element method using 1-D finite element mesh to efficiently compute the lower-order wave types and damping loss factors of a viscoelastic laminate. He formulated the cross-sectional displacement field of the laminate and the characteristic equation for free-wave propagation as a linear algebraic eigenvalue problem in wave number. Plagianakos and Saravanos [21] presented an integrated high-order layerwise formulation and a 2-node damped beam element for predicting the damped free-vibration and thick composite sandwich beams, for which quadratic and cubic fields are added to the linear layerwise formulation in the kinematics of each discrete layer while maintaining displacement compatibility. Later, they developed a 4-node C1 continuous damped plate element by extending their previous work and predicted the damped dynamic characteristics of thick composite and sandwich composite plates [22]. Moreira et al. [23] developed a 4-node quadratic facet-shell finite element, based on a generalized layerwise formulation, to simulate multiple viscoelastic layer or multiple soft core sandwich plates. They also adopted the MITC approach to protect shear locking and the drilling degrees of freedom to protect the usual membrane locking of low order facet-cell elements.

More recently, Amichi and Attalla [24] introduced a damped beam element with 18 DOFs for three-layer curved symmetric and unsymmetric sandwich beams with a viscoelastic core based upon the discrete displacement approach. The in-plane and transverse displacements are interpolated with C0 continuous linear and cubic polynomials, respectively, and the rotational influence of the transversal shearing in the core. Chinnet and Attalla [25] introduced an analytical discrete laminate method to model thick composite laminate and sandwich plates and beams with linear viscoelastic damping layers, which can handle symmetric and asymmetric layouts of unlimited number of transversal incompressible layers. In their DLM approach making use of the first order shear deformation theory, each layer was modeled as thick laminate with orthotropic orientation, rotational inertia and transversal shearing, membrane and bending deformation. Assaf [26] extended his previous displacement-based FE formulation for three-layer damped sandwich plates to sandwich beams made up of cross-ply laminate faces with arbitrary number of 0° and 90° plies and a viscoelastic core. The formulation was based on a layerwise linear axial displacement through the beam thickness and a 2-node 8-DOF damped beam element was developed using Lagrange linear functions for the mean and relative axial displacements and Hermite cubic functions for the transverse displacement.

Meanwhile, Mead and Markus’s six-order differential equation governing the forced vibration of the three-constrained-layer damped beams is expressed in terms of only the transverse displacement, so that it provides one a clear understanding of the problem at hand, especially when a transverse external loading is applied to the damped beam. But, there exists a difficulty when one tries to approximate it using the finite element method, because the six-order differential equation expressed in terms of the lateral and axial displacements inherently requires C2-basis functions [27]. On the other hand, the alternative Mead and Markus’s two coupled equations [6], which require a mixed finite element approximation, suffer from the numerical difficulty in dealing with various boundary conditions of multi-layered damped beam structures.

Based upon the literature review and the numerical difficulty in implementing Mead and Markus’s differential equations, this paper intends to introduce a 2-node damped beam element according to the Mead and Markus’s approach in order for the DOF-efficient forced vibration analysis of three-layered symmetric straight sandwich beams with a viscoelastic core. Motivated by the fact that the longitudinal displacements of two homogeneous faces can be replaced with the transverse shear of the core layer, a 2-node damped beam element with only three DOFs per node is developed by taking the variation of the virtual kinetic and strain energies expressed in terms of the lateral displacement of beam and the transverse shear strain of a core layer. These state variables are interpolated with element-wise polynomials, and the C0-basis functions are analytically derived using the compatibility relation between two. Meanwhile, the forced vibration equations of three-constrained-layer damped beam must be equipped with three pairs of boundary conditions so that the rotation of the mid-surface which is directly derived from the lateral displacement is added. The characteristics of the proposed damped beam element with respect to the element number, the beam slenderness, the thickness of core layer and the boundary condition treatment are investigated through the numerical experiments. As well, the applicability to sandwich beam with a frequency-dependent viscoelastic core and the DOF-efficiency of the proposed damped beam element are verified from the comparison with the results obtained by 3-D Nastran solid elements and other authors.

2. Forced vibration of three-layered damped sandwich beam

2.1. Governing equations based on the beam theory

Referring to Fig. 1, let us consider a three-layered sandwich beam of length L and unit width with rectangular cross-section subject to a time-dependent vertical loading q(t). The upper and lower face layers of thicknesses h1 and h3 are purely linear elastic with Young’s moduli of E1 and E3. While, a core layer of thickness h2 is linearly viscoelastic with a complex shear modulus
$G^* = G(1 + i\eta)$ in which $\eta$ denotes the loss factor. Throughout this paper, $(-)^*$ indicate the complex values. The total thickness of the beam is sufficiently small compared to the beam length, and three layers are assumed to be completely bonded such that no slipping occurs at the interfaces.

Further assumptions for the analysis of the beam transverse displacement are as follows: (1) the transverse direct strains in both the core and face layers are small enough so that the lateral displacements of three layers are uniform across any section of the beam, (2) the longitudinal and rotary inertia effects are ignorable, (3) the face layers obey Kirchhoff hypothesis so that no transverse shear strain occurs, (4) the core layer obeys Kerwin assumption [5] that the longitudinal direct strain in the core layer is much smaller than the shear strain, and (5) the numerical results at a given constant temperature are presented in this paper.

The complex transverse shear stress in the core layer in which the longitudinal displacement $u$ is uniform through the thickness is $\tau^* = G^*\gamma = G^*(\partial u/\partial z + \partial w/\partial x)$. Here, the slope of $u$ with respect to the $x$-axis is found to be

$$\frac{\partial u}{\partial z} = \frac{1}{h_2} \left[ (u_1 + h_1 \frac{\partial w}{\partial x}) - (u_3 - h_3 \frac{\partial w}{\partial x}) \right] = \frac{(h_1 + h_3) \frac{\partial w}{\partial x} + u_1 - u_3}{h_2},$$

from the deformed configuration shown in Fig. 2. Substituting Eq. (1) into the definition of complex shear stress leads to

$$\tau^* = G^* \left[ \frac{d \partial w}{\partial x} \left( \frac{u_1 - u_3}{h_2} \right) \right].$$

One can confirm that the shear strain and stress in the core layer are uniform across the thickness, which is consistent with the above mentioned assumption (4).

Letting the flexural rigidities of two face layers be $D_1 = E_h h_1^3/12$ and $D_2 = E_h h_2^3/12$, the shear resultants exerted on two face layers are calculated by

$$S_1 = D_1 \frac{\partial^2 w}{\partial x^2}, \quad S_3 = D_1 \frac{\partial^2 w}{\partial x^2},$$

The complex resultant shear force exerted on the core layer is $S_2^* = -\tau^* d$, so the total complex shear force produced on the sandwich beam cross-section becomes

$$S_2^* = (D_1 + D_3) \frac{\partial^2 w}{\partial x^2} - G^* d \left[ \frac{d \partial w}{\partial x} \left( \frac{u_1 - u_3}{h_2} \right) \right].$$

Viewing the sandwich beam subject to the distributed transverse dynamic load $q(x); t$ as an Euler beam and using the relation of $\partial S_2/\partial x = q - \rho \partial^2 w/\partial t^2$, one can derive the following fourth-order differential equation for the dynamic displacements $w(x); t = W(x)e^{i\omega t}u_1(x); t = U_1(x)e^{i\omega t}$ and $u_3(x); t = U_3(x)e^{i\omega t}$ given by

$$\frac{d^4}{dx^4} W - \frac{G^* d}{h_2 D_1} \left[ \frac{d^2}{dx^2} \left( \frac{dU_1}{dx} - \frac{dU_3}{dx} \right) - \rho \omega^2 \frac{W}{D_1} \right] = \frac{Q}{D_1},$$

with $D_1 = D_1 + D_3$ and the mass density $\rho$ of the sandwich beam.

Meanwhile, from the elementary mechanics relation, the derivation of the axial resultant forces $P_1$ and $P_2$ in each face layer is straightforward:

$$P_1 = E_h h_1 \frac{\partial u_1}{\partial x}, \quad P_3 = E_h h_3 \frac{\partial u_3}{\partial x},$$

The force equilibrium in the axial direction gives rise to the constant given by

$$P_1 + P_3 = 0, \quad E_h h_1 \frac{\partial u_1}{\partial x} + E_h h_3 \frac{\partial u_3}{\partial x} = 0.$$

Using Eq. (7), together with the relation of $\partial u_1/\partial x = -\partial u_3/\partial x$, one can rewrite Eq. (5) as

$$\frac{d^2}{dx^2} \left( \frac{d^2 U_1}{dx^2} - \frac{G^*}{E_h h_1} + \frac{G^*}{E_h h_3} \right) U_3 = \frac{G^* d}{E_h h_2} \frac{1}{D_1} \frac{d W}{D_1}.$$

The force equilibrium of an element $dx$ of face layer (3) in the axial direction ends up with $\tau = -\partial P_3/\partial x$. Plugging this relation into Eq. (2), together with the relation between $u_3$ and $P_3$ in Eq. (6) and one between $u_3$ and $U_3$ in Eq. (7), leads to the second-order differential equation for the longitudinal displacement $u_3$ given by

$$\frac{d^2}{dx^2} U_3 - \frac{G^*}{E_h h_3} \frac{1}{D_1} \frac{d W}{D_1} = \frac{G^* d}{E_h h_2} \frac{1}{D_1} \frac{d W}{D_1}.$$

Introducing two symbols $Y = d^2(E_h h_1 E_h h_3)/(D_1(E_h h_1 + E_h h_3))$ and $g^* = G^* d^2 h_3 D_1 Y$ into Eqs. (8) and (9), one can get Mead and Markus's two coupled equations [6]:

$$\frac{d^2}{dx^2} W = g^* \frac{d^2}{dx^2} \frac{d^2}{dx^2} W - \rho \omega^2 \frac{W}{D_1} + g^* \frac{E_h h_3}{D_1} \frac{d U_3}{D_1} = \frac{Q}{D_1}$$

and a six-order differential equation given by

$$\frac{d^6}{dx^6} W = \rho \omega^2 \frac{d^2}{dx^2} \frac{d^2}{dx^2} W - \rho^2 \frac{d^6}{dx^6} W + g^* (1 + Y) \frac{d^2}{dx^2} W + g^* \frac{\rho^2}{D_1} \frac{d^2}{dx^2} W = \frac{1}{D_1} \frac{d^2 Q}{dx^2} - g^* Q.$$

Note that Eq. (12) can be derived according to the numerical manipulation using Eqs. (10) and (11) and their differentiations. Furthermore, $g^* = G^* d^2 h_2 Y (1 + i\eta) = g(1 + i\eta)$, and the boundary conditions for two different governing equations are given in Appendix B.

### 3. Three-field damped beam element

A difficulty in the finite element approximation of the variational formulation (12) is the use of basis functions belonging to the function space $H^4(0, L)$ [27]. In order words, double differentiations of each basis function should be continuous at the common interfaces between elements. Meanwhile, the variational formulation of Mead and Markus's two coupled Eqs. (10) and (11) results in a mixed form, and furthermore their physical interpretation is rather complex to derive the shape functions and the element-wise matrices using the energy theory. Additionally, one may encounter the numerical difficulty in dealing with various boundary conditions of multi-layered damped beam structures. In this context, we replace the longitudinal displacement $u_3$ with the transverse shear strain $\gamma(x; t) = \Gamma(x)e^{i\omega t}$ using the relation of $u_3 = (\partial w / \partial z) / 2$ which is derived from Eq. (2) for the symmetric damped sandwich structures.
3.1. Forced vibration in terms of $w$, $\theta$ and $\gamma$

Referring to Figs. 1 and 2, the total energy stored within the symmetric damped sandwich beam becomes a sum of the kinetic energy $T$,

$$T = \frac{1}{2} \int_0^L \rho \left( \frac{\partial w}{\partial t} \right)^2 \, dx$$

and the strain energy $V_S$ and the extensional strain energy $V_E$ of two face plates and the shear strain energy $V_S$ of the core layer,

$$V_S = \frac{1}{2} \int_0^L D_1 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx$$

$$V_E = \frac{1}{2} \int_0^L \left[ E_1 h_1 \left( \frac{\partial u_1}{\partial x} \right)^2 + E_2 h_2 \left( \frac{\partial u_2}{\partial x} \right)^2 \right] \, dx = \frac{1}{2} \int_0^L Y_D \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{\partial^2 \psi}{\partial x^2} \right)^2 \, dx$$

$$V_S = \frac{1}{2} \int_0^L C_2 \left( \frac{\partial e_2}{\partial x} \right)^2 \, dx$$

With $\omega^{ext}(t)$ being the work done by the external load, for example, the distributed load $q(x,t)$, the total net strain energy $\Pi(t)$ at time $t$ is defined by $\Pi(t) = V(t) + W^{ext}(t)$. Then, the Lagrangian functional $L(t)$ at time $t$ is defined by

$$L(w,w,x;\gamma,\gamma) = T(w) - \Pi(w,w,x;\gamma,\gamma)$$

The generalized Hamilton principle for a non-conservative damped sandwich structure during a time period $t$ of observation is given by

$$\delta \int_0^t L \, dt = \delta \int_0^t (T-V+W^{ext}) \, dt = 0$$

One can easily derive the governing equations in terms of $W$ and $\Gamma$ [28]:

$$\rho \rho^2 \frac{d^2 W}{dt^2} + D_1 \frac{d^2 \Gamma}{dx^2} - D_1 Y \frac{d^3 \Gamma}{dx^3} = 0$$

and three boundary conditions given by ($N_i$ being the axial resultant force)

$$N_1 = D_1 \left( 1 + Y \right) \frac{d^2 W}{dx^2} - Y \frac{d^3 \Gamma}{dx^3}$$

$$N_2 = -D_1 \left( 1 + Y \right) \frac{d^2 W}{dx^2} - Y \frac{d^3 \Gamma}{dx^3}$$

The above governing equations and boundary conditions could be also derived by plugging the relation of $u_3=(dw-h_2)/2$ into Eqs. (10) and (11) and into Eqs. (B1)-(B3) in Appendix B.

3.2. Derivation of shape functions

Fig. 3 shows a portion of beam with two nodes, where each node has three degrees of freedom $(w, \theta, \gamma)^T$. Note that the rotation $\theta$ of the neutral axis of sandwich beam is added to fulfill the three essential boundary conditions. Each beam element discretized for the damped sandwich structure is mapped onto the master element with the length of 2. The dynamic transverse displacement $w(x,t)$ is expressed as a linear combination of monomials such that

$$w(x,t) = z_1^2(t) + z_2^2(t) + z_3^2(t) + z_4^2(t) + z_5^2(t) + z_6^2(t)$$

where $\{p_i^2(t)\} = \{z_1^2(t) z_2^2(t) z_3^2(t) z_4^2(t) z_5^2(t) z_6^2(t)\}^T$. In the similar manner, the transverse shear strain $\gamma(x,t)$ is expressed by

$$\gamma(x,t) = b_1^2(t) + b_2^2(t) + b_3^2(t)$$

with $\{p_i^2(t)\} = \{z_1^2(t) z_2^2(t) z_3^2(t) z_4^2(t) z_5^2(t) z_6^2(t)\}^T$. Note that $z_i^2(t)$ and $b_i^2(t)$ are complex.

Substituting Eqs. (24) and (25) into Eq. (20) and solving the resulting linear complex equation system ends up with

$$\{b_i(t)\} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1^2(t) \quad z_2^2(t) \quad z_3^2(t) \quad z_4^2(t) \quad z_5^2(t) \end{bmatrix}$$

with $c_i^2 = -6d_1/\alpha^2 g^2 h_2$ and $c_2^2 = -120d_1/\alpha^2 (g^4) h_2$. And the rotation of the neutral axis is defined by

$$\theta(x,t) = \omega \frac{\partial \gamma^2}{\partial x^2} = \frac{1}{a} \left[ z_1^2(t) + 2z_2^2(t) + 2z_3^2(t) + 2z_4^2(t) + 2z_5^2(t) + 2z_6^2(t) \right]$$

Six degrees of freedom of each damped sandwich beam can be expressed in the following complex matrix form:

$$\{w\} = \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} = \begin{bmatrix} \alpha \{1, 1, 1, 1, 1, 1, 1, 1, 1\} \end{bmatrix} \begin{bmatrix} \{z_1^2(t)\} \end{bmatrix}$$

Furthermore, we have

$$w(x,t) = \{P(t)\} \{z^2(t)\} = \{P(t)\} \{D^*\} \{w\} = \{N^*\} \{\gamma\}$$

Finally, one can derive the $(1 \times 6)$ complex matrix $[N^*\{\gamma\}]$ composed of six complex shape functions $N_i^*\{\gamma\}$ defined by

$$N_1^*\{\gamma\} = \frac{c_1^2(-2z_3^2-\gamma^2)+c_2^2(-16+\gamma+\gamma^2)}{4c_1^2+c_2^2}$$

$$\cdots$$

$$N_6^*\{\gamma\} = \frac{c_5^2(-2z_3^2-\gamma^2)+c_6^2(-16+\gamma+\gamma^2)}{4c_1^2+c_2^2}$$

$$N_7^*\{\gamma\} = \frac{c_1^2(-2z_3^2-\gamma^2)+c_2^2(-16+\gamma+\gamma^2)}{8c_1^2+c_2^2}$$

$$N_8^*\{\gamma\} = \frac{c_1^2(-2z_3^2-\gamma^2)+c_2^2(-16+\gamma+\gamma^2)}{4c_1^2+c_2^2}$$
and the element-wise load vector, the variations of the kinetic and grand variations using a commercial program MATHEMATICA.

The frequency responses obtained using the damped beam elements and 3-D Nastran solid elements for four different element densities along the beam neutral axis are compared in

Note that \( \lambda^2 \) in Eq. (45) stands for \( \lambda^2 \). Summing up the element-wise variations over the entire finite element mesh leads to

for a sinusoidal external force \( q(x,t) = \bar{q}(x) \) and the dynamic response \( \mathbf{W}(t) = \mathbf{W} \mathbf{e}^{i \omega t} \). Note that all the real and imaginary parts of matrices and load vectors are in function of the loss factor \( \eta \) because the loss factor \( \eta \) is included in the complex shape functions and Eq. (45).

Rewriting Eq. (47) into the frequency domain ends up with the linear matrix equations for solving the complex eigenvalues:

with the damping matrix \( \mathbf{C} = \frac{1}{\omega} [\mathbf{K}_m(\eta)] - \omega \mathbf{M}_m(\eta) \). In addition, the frequency response \( \mathbf{W} \) to a given excitation frequency \( \omega_{ex} \) is calculated by

where

4. Numerical experiments

A test FEM program was coded according to the numerical formulas described in Section 3, for which the direct solver for complex matrices was employed to solve the damped frequency responses. Five numerical examples are considered to demonstrate the validity and the DOF-efficiency of the proposed three-layered damped beam element. Rubber is taken for a core layer and steel is for two face layers, and the corresponding material properties are given in Table 1, where the loss factor \( \eta \) is taken variable for the parametric investigation. The frequency responses of three model problems were also obtained by 8-node 3-D Nastran solid elements for the comparison purpose.

Fig. 4 shows a three-layered damped sandwich beam subject to the vertical unit impulse at the center of the beam neutral axis, where the beam dimensions are as follows: \( L = 320 \text{ mm} \), \( b = 9 \text{ mm} \), and \( h_1 = h_2 = h_3 = 3 \text{ mm} \). Respectively, the left and right ends of the beam are clamped \( w = \theta = \gamma = 0 \) and simply supported \( w = 0 \), respectively, and the dynamic response is taken at the point 200 mm from the left end where the unit impulse is applied. The neutral axis of the beam is uniformly divided to generate both the damped beam and 3-D Nastran solid models, and furthermore the beam cross-section is discretized with 81 elements for 3-D Nastran solid model \( (3+3+3 \text{ in the thickness direction and } 9 \text{ in the width direction}) \). With this model, the convergence of lowest peaks in the frequency response to the element number along the neutral axis is examined.

The frequency responses obtained using the damped beam elements and 3-D Nastran solid elements for four different element densities along the beam neutral axis are compared in

![Fig. 4. A clamped and simply supported three-layered damped sandwich beam (unit:mm).](image-url)
Fig. 5. The beam model and the 3-D Nastran solid model show the remarkable discrepancy in the frequency responses when the element density is not sufficiently high, and the discrepancy becomes larger in proportion to the frequency. However, it is clearly observed that the difference between the damped element and 8-node Nastran solid element is negligible up to the eighth peak when the neutral axis discretized into 64 elements. Thus, it is confirmed that the proposed damped beam element provides the reliable damped frequency response of the three-layered clamped-simply supported sandwich beam, when the neutral axis is discretized into a reasonable number of elements.

The detailed numerical values of the resonance frequencies and receptances of lowest five peaks with respect to the element number are given in Table 2. When compared with 3-D Nastran solid element, the proposed damped beam element shows much more rapid convergence in the resonance frequencies and receptances. The resonance frequency shows the convergence at 8 elements for the first and second peaks and at 32 elements for the third, forth and fifth peaks. While, in the receptance, the first, second and third peaks show the convergence at 32 elements but the forth and fifth peaks show the convergence at 64 elements. One can observe that the 3-D Nastran solid model requires more elements to obtain the converged resonance frequencies and receptances. The excellent convergence of the proposed damped beam element becomes more apparent when additional element numbers required to discretize the beam cross-section of 3-D Nastran solid model is considered. The maximum relative errors at the fifth peak between two element types are 0.52% in the resonance frequency and 2.24% in the receptance, respectively.

Next, we consider a cantilever sandwich beam shown in Fig. 6 to investigate the variation of frequency response to the beam slenderness $L/h$ ($h=h_1+h_2+h_3$). The clamped boundary condition ($w=\theta=\gamma=0$) is specified to the left end. The beam width $b$ and the thicknesses $h_i$ of three layers and the material properties of steel and rubber are the same as for the previous problem. Meanwhile, the beam length $L$ is taken variable such that $L/h$ becomes 4, 8, 16 and 32, and the beam is discretized along the neutral axis with the uniform 16 elements per $L$ of 100 mm. The cross-section of the 3-D Nastran solid model is discretized in the same manner as for the previous problem. The damped dynamic response is evaluated at the right end of the beam neutral axis where unit impulse is applied.

The variation of frequency response to the beam slenderness $L/h$ is represented in Fig. 7, for which the beam length $L$ is taken variable with the total beam height $h$ fixed by 9 mm. The damped beam model and the 3-D Nastran solid model show the remarkable difference when $L/h$ is 4, but it is clearly observed that the difference monotonically decreases in proportion to the beam slenderness. It is consistent well with the fact that the 3-D elasticity model approaches the Euler beam theory as the relative thickness becomes smaller [8,9]. In other words, the Euler beam theory serves as a limit model for 3-D full elasticity problems with thin domain. It is observed from Fig. 7 that the difference between the damped beam model and the 3-D Nastran solid...
model becomes negligible up to the eighth peak when the beam slenderness is 32, confirming that the proposed beam element provides the consistent damped frequency response to the beam slenderness. The detailed numerical values of the resonance frequencies and receptances are recorded in Table 3. One can observe the uniform decrease in the differences of the resonance frequencies and receptances in proportion to the beam slenderness for all the lowest peaks. The maximum relative errors at \( L/h = 32 \) between two element types are 0.27% in the resonance frequency and 5.88% in the receptance, respectively.

Fig. 8 shows the variation of frequency response to the relative thickness \( h_2/h \times 100\% \) of the core viscoelastic layer to the total beam height. Referring to the previous two damped sandwich beam models, the dimensions taken for this example are \( L = 320 \text{ mm} \) and \( b = h = 9 \text{ mm} \) and the beam neutral axis is uniformly discretized with the mesh density equal to 16 elements per \( L = 100 \text{ mm} \). The left and right ends of beam are simply supported \( (w = 0) \) and a unit impulse is applied at the point 20 mm from the left end. The dynamic response is evaluated at the point where the unit impulse is applied. It is observed that the resonance frequencies and receptances become smaller when a viscoelastic layer is inserted. The detailed numerical values of lowest three resonance frequencies and their receptances are given in Table 4 for six different relative thicknesses of the core. All the resonance frequencies decrease with the increase of relative thickness up to \( h_2/h \) of 30%, but the first and second resonance frequencies increase thereafter. The receptances increase with the relative thickness except for those of the first and second resonance frequencies when the core is very thin. Thus, it is desired to restrict the core thickness to thin for higher damping performance.

In order to investigate the variation of frequency response to the implementation method of the clamped boundary condition, the first numerical example shown in Fig. 4 was analyzed again for three different boundary conditions: \( w = \theta = 0, w = \gamma = 0 \) and \( w = \theta = \gamma = 0 \) at the left end. The geometry dimensions and material properties, the mesh density and the simply supported boundary at the right end except for the beam length \( L \) and the location of a unit impulse are kept unchanged. The beam length \( L \) is changed from 320 mm to 400 mm and the unit impulse is applied at the center of the beam. The dynamic response is evaluated at the point where the unit impulse is applied. The condition of \( w = \theta = 0 \) implies that the beam neutral axis is vertically supported and the top and bottom faces are bolted, while \( w = \gamma = 0 \) indicates that the beam neutral axis is vertically supported and only the side of core is completely bonded. Finally, \( w = \theta = \gamma \) implies that the beam is vertically supported and the side of core is bonded and the sides of the top and bottom faces are bolted.

It is observed from Fig. 9 that three cases show the remarkable difference in the resonance frequencies and receptances at the first peak, but the difference between \( w = \theta = 0 \) and \( w = \theta = \gamma = 0 \) becomes smaller as the peaks go higher. Thus, the influence of bonding of the core side is significant only at the first resonance frequency. Meanwhile, the difference between \( w = \gamma = 0 \) and \( w = \theta = \gamma = 0 \) does not diminish but remain even at higher peaks, and furthermore the case of \( w = \gamma = 0 \) does not show the clear resonance peaks near 400 Hz and 1000 Hz. Therefore, the bonding of the sides of two faces significantly influences the overall frequency response of the damped sandwich beam. From the detailed numerical values of resonance frequencies given in Table 5, one can clearly confirm that \( w = \theta = 0 \) and \( w = \gamma = 0 \) are the relaxed constraints of \( w = \theta = \gamma = 0 \) and \( w = \gamma = 0 \) is the weakest constraint. This apparent physical difference is also clearly observed from the comparison of the receptances. The case of \( w = \gamma = 0 \) produces the receptances which are significantly different from those of the remaining two cases. It is convinced from this experiment that the use of \( w, \theta \) and \( \gamma \) as the nodal DOFs can successfully distinguish three different clamped boundary conditions of three-layered damped sandwich beam.

Next, a free-free symmetric sandwich beam with a moderately thick and low loss factor viscoelastic core which was dealt in a paper by Amichi and Atalla [24] is considered, in order to verify the proposed damped sandwich beam element when the viscoelastic core has frequency-dependent material properties. Referring to Fig. 4, the dimensions of the beam are as follows: \( L = 0.46038 \text{ m} \), \( b = 0.05108 \text{ m} \) and \( h_1 = h_2 = h_3 = 0.635 \text{ mm} \), and the material properties are given in Table 6 where the shear modulus of the viscoelastic core as well as the damping coefficients are frequency dependent. The frequency response is taken at midspan where a harmonic point force is applied. The beam axis is uniformly discretized with 50 elements based on the previous convergence test to the element number.

A comparison of the driving point impedance predicted by the proposed damped sandwich beam with the numerical result obtained by Amichi and Atalla [24] and experimental data of Sun and Lu [29] is shown in Fig. 10. It can be observed that the two results predicted by the proposed beam element and by Amichi and Atalla [24] are in excellent agreement. Also, it is observed that

<table>
<thead>
<tr>
<th>Peaks Number of elements</th>
<th>Residence frequencies (Hz)</th>
<th>Receptances (m/N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present Nastran Present Nastran</td>
<td>Present Nastran Present Nastran</td>
<td></td>
</tr>
<tr>
<td>1 8 221 227 6.8282E – 05 6.6735E – 05</td>
<td>128 221 221 6.8338E – 05 6.8352E – 05</td>
<td></td>
</tr>
<tr>
<td>16 221 223 6.8333E – 05 6.7785E – 05</td>
<td>32 221 221 6.8338E – 05 6.8161E – 05</td>
<td></td>
</tr>
<tr>
<td>64 221 221 6.8338E – 05 6.8352E – 05</td>
<td>128 221 221 6.8338E – 05 6.8394E – 05</td>
<td></td>
</tr>
<tr>
<td>(0.01%) (0.01%)</td>
<td>(0.03%) (0.03%)</td>
<td></td>
</tr>
<tr>
<td>128 557 556 3.4752E – 06 3.4271E – 06</td>
<td>(0.18%)</td>
<td>(0.14%)</td>
</tr>
<tr>
<td>3 8 978 1150 1.4017E – 06 1.7240E – 06</td>
<td>16 977 1016 1.3920E – 06 1.4982E – 06</td>
<td></td>
</tr>
<tr>
<td>32 976 985 1.3913E – 06 1.4471E – 06</td>
<td>64 976 977 1.3913E – 06 1.4318E – 06</td>
<td></td>
</tr>
<tr>
<td>128 976 974 1.3913E – 06 1.4276E – 06</td>
<td>(0.21%) (2.54%)</td>
<td></td>
</tr>
<tr>
<td>4 8 1505 2105 2.0077E – 06 2.2870E – 06</td>
<td>16 1501 1615 1.9778E – 06 2.0197E – 06</td>
<td></td>
</tr>
<tr>
<td>32 1500 1525 1.9757E – 06 1.9676E – 06</td>
<td>64 1500 1502 1.9756E – 06 1.9582E – 06</td>
<td></td>
</tr>
<tr>
<td>128 1500 1496 1.9756E – 06 1.9574E – 06</td>
<td>(0.27%) (0.93%)</td>
<td></td>
</tr>
<tr>
<td>128 2901 2886 6.6220E – 07 6.7734E – 07</td>
<td>(0.52%) (2.24%)</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 6. A three-layered damped cantilever sandwich beam.
the three results are in good agreement for the first three modes, and the discrepancies observed above 1000 Hz which may be due to the tests were also reported by Sun and Lu [29]. As briefly summarized in the introduction, the three-layer damped beam element of Amichi and Atalla was formulated based on the discrete displacement approach by selectively adopting a Timoshenko hypothesis for the viscoelastic core and Euler–Bernoulli hypotheses for the elastic faces. It accounts for the rotational influence of the transversal shearing core, and the coordinate transformation is introduced at the level of element-wise stiffness and mass matrices. It can be applicable to curved sandwich beams as well as to unsymmetrical configurations, but the total number of DOFs per element reaches 18. On the other hand, the three results are in good agreement for the first three modes, and the discrepancies observed above 1000 Hz which may be due to the tests were also reported by Sun and Lu [29]. 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### Table 3

Resonance frequencies and receptances to the beam slenderness $L/h$.

<table>
<thead>
<tr>
<th>Peaks $L/h$</th>
<th>Resonance frequencies (Hz)</th>
<th>Receptances (m/N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
<td>Nastran</td>
</tr>
<tr>
<td>1</td>
<td>4.2279</td>
<td>2.283</td>
</tr>
<tr>
<td>2</td>
<td>12.010</td>
<td>11.729</td>
</tr>
<tr>
<td>3</td>
<td>32.490</td>
<td>22.4108</td>
</tr>
<tr>
<td>4</td>
<td>32.490</td>
<td>32.490</td>
</tr>
<tr>
<td>5</td>
<td>104.028</td>
<td>34.544</td>
</tr>
<tr>
<td>6</td>
<td>63.136</td>
<td>30.500</td>
</tr>
<tr>
<td>7</td>
<td>4.369</td>
<td>4.356</td>
</tr>
<tr>
<td>8</td>
<td>4.369</td>
<td>4.356</td>
</tr>
<tr>
<td>9</td>
<td>1.346</td>
<td>1.348</td>
</tr>
<tr>
<td>10</td>
<td>104.028</td>
<td>34.544</td>
</tr>
<tr>
<td>11</td>
<td>63.136</td>
<td>30.500</td>
</tr>
<tr>
<td>12</td>
<td>4.369</td>
<td>4.356</td>
</tr>
<tr>
<td>13</td>
<td>4.369</td>
<td>4.356</td>
</tr>
<tr>
<td>14</td>
<td>1.346</td>
<td>1.348</td>
</tr>
</tbody>
</table>

Fig. 7. Variation of FRF to the beam slenderness $L/h$: (a) 4, (b) 8, (c) 16, and (d) 32.

Fig. 8. Dependence of FRF on the relative thickness of a core layer.
hand, the proposed three-layer damped beam element was formulated by applying the Mead and Markus's concept to the virtual work principle. The proposed damped beam element is restricted to the symmetric straight configurations, but it has only three DOFs per node (i.e., six DOFs per element).

Table 7 compares the total numbers of elements and degrees of freedom which are used to analyze the cantilever beam structure shown in Fig. 6 when \( L/h \) is 32 and \( h \) is 9 mm. The total number of beam elements is calculated from the fact that the beam neutral axis is discretized with the uniform 16 elements per \( L \) of 100 mm. Note that the total discretization numbers along the beam neutral axis are the same for both the beam and 3-D Nastran solid models. Even though the comparison is made based on only this small-scale simple beam structure, the DOF-efficiency of the proposed damped beam element has been clearly justified. Furthermore, there is no doubt that the DOF-efficiency becomes more apparent when the element number required to discretize the beam cross-section for more complicated 3-D solid models is considered.

### Table 4
Resonance frequencies and receptances to the relative thickness of a core layer.

<table>
<thead>
<tr>
<th>Relative thickness (%)</th>
<th>Resonance frequencies (Hz)</th>
<th>Receptances (m/N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>First peak</td>
<td>Second peak</td>
</tr>
<tr>
<td>0</td>
<td>206 825</td>
<td>1855</td>
</tr>
<tr>
<td>3</td>
<td>200 726</td>
<td>1469</td>
</tr>
<tr>
<td>7</td>
<td>195 668</td>
<td>1309</td>
</tr>
<tr>
<td>10</td>
<td>190 618</td>
<td>1189</td>
</tr>
<tr>
<td>30</td>
<td>175 605</td>
<td>923</td>
</tr>
<tr>
<td>50</td>
<td>176 667</td>
<td>801</td>
</tr>
<tr>
<td>70</td>
<td>179 672</td>
<td>771</td>
</tr>
</tbody>
</table>

Fig. 9. Frequency responses for three different boundary conditions.
core viscoelastic layer. The lateral deflection and the transverse shear strain were approximated by fifth- and third-order monomials and the rotation is defined by the direct differentiation of the lateral deflection. Each element has two nodes and three state variables were defined as the degrees of freedom per node, and the corresponding six real and six imaginary shape functions were derived based on the compatibility relation between the lateral displacement and the transverse shear strain.

The validity and convergence and the DOF-efficiency and applicability to frequency-dependent sandwich beam of the proposed damped beam element were examined through the benchmark experiments. According to our investigation through the numerical experiments, the following main observations are drawn.

1. The proposed damped beam element shows more rapid convergences in the resonance frequencies and receptances than 3-D Nastran solid element, when measured in terms of the total element number along the beam neutral axis.
2. The differences of the resonance frequencies and receptances between the proposed damped beam element and 3-D Nastran solid element become smaller as the beam relative thickness decreases.
3. The variation of the frequency response to the relative thickness of a core viscoelastic layer is successfully analyzed by the proposed damped beam element. Furthermore, the proposed damped beam element successfully analyzes the differences among three different boundary conditions; \( w = \gamma = 0 \), \( w = \theta = 0 \) and \( w = \theta = \gamma = 0 \).

4. It has been justified, from the benchmark test of the sandwich beam with a frequency-dependent viscoelastic core, that the numerical result predicted by the proposed damped beam element shows an excellent agreement with those of other authors.
5. The DOF-efficiency of the proposed damped beam element has been justified such that the frequency response of the damped sandwich beam structures can be effectively analyzed with the extremely small number of elements, when compared with 3-D solid element.

However, the proposed damped beam element leaves a further refinement and extension for the forced vibration analysis of the damped asymmetric curved beam structures in which the torsional strain energy can not be ignorable, and which represents a topic that deserves future work.

Acknowledgement

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(ADD) under Contract No. UD03000AD. This work was supported by the Human Resources Development of the Korea Institute of Energy Technology Evaluation and Planning (KETEP) grant funded by the Korea Ministry of Knowledge Economy (No. 20113020020010). The financial support for this work through World Class 300 from Ministry of Knowledge Economy of Korea.

Fig. A5. Complex shape function \( N_0(z) \): (a) real and (b) imaginary.

Fig. A6. Complex shape function \( N_1(z) \): (a) real and (b) imaginary.

Fig. A7. Complex shape function \( H_1(z) \): (a) real and (b) imaginary.

Fig. A8. Complex shape function \( H_2(z) \): (a) real and (b) imaginary.

Fig. A9. Complex shape function \( H_3(z) \): (a) real and (b) imaginary.

Fig. A10. Complex shape function \( H_4(z) \): (a) real and (b) imaginary.
Appendix A. Complex shape functions $N_1^*(\zeta) \sim N_6^*(\zeta)$ and $H_1^*(\zeta) \sim H_6^*(\zeta)$

The real and imaginary parts of the complex shape functions are calculated with the material properties given in Table 1 and the geometry dimensions set by $a=1\,\text{mm}$, $d=6\,\text{mm}$ and $h_1=h_2=h_3=3\,\text{mm}$. (See Figs. A1–A12)

Appendix B. Boundary conditions

One can derive the boundary conditions for Mead and Markus's governing equations for the damped forced vibration by taking the symmetric variational formulation.

- Mead and Markus's two coupled equations:
  
  \[
  S_i = D_i \left( \frac{d^2 W}{dx^2} - g^* (1 + Q) \frac{d^2 W}{dx^2} - \frac{\rho_0 c^2}{D_i} \frac{dW}{dx} \right) \quad \text{or} \quad W
  \]
  
  \[
  M_i = -D_i \frac{d^2 W}{dx^2} \quad \text{or} \quad W
  \]
  
  \[
  N_i = E_i h_i \frac{dU_3}{dx} \quad \text{or} \quad U_3
  \]

- Mead and Markus's six-order differential equation [6]:
  
  \[
  S_i = D_i \left( \frac{d^3 W}{dx^3} + g^* (1 + Q) \frac{d^3 W}{dx^3} - \frac{\rho_0 c^2}{D_i} \frac{dW}{dx} \right) \quad \text{or} \quad W
  \]
  
  \[
  M_i = D_i \left( \frac{d^4 W}{dx^4} + g^* (1 + Q) \frac{d^4 W}{dx^4} + Q \right) \quad \text{or} \quad W
  \]
  
  \[
  S_i = D_i \frac{d^3 W}{dx^3} \quad \text{or} \quad W''
  \]

Note that $S_i = S_1 + S_2 + S_3$ is the shear force resultant defined in Eq. (4), $M_i = M_1 + M_2$ and $S_j = S_1 + S_2$ are the bending moment and shear force resultants exerted on only two face plates, respectively.

References